A RESULT ABOUT PICARD-LEFSCHETZ MONODROMY

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Abstract. Let f and g be reduced homogeneous polynomials in separate sets of variables. We establish a simple formula that relates the eigenspace decomposition of the monodromy operator on the Milnor fiber cohomology of fg to that of f and g separately. We use a relation between local systems and Milnor fiber cohomology that has been established by D. Cohen and A. Suciu.

1. Brief Introduction and Statement of Results

Thom-Sebastiani type Theorems have a rich history. This study was initiated by Sebastiani and Thom [12] and improved by others [8, 11]. Perhaps the most successful generalization has been achieved by Némethi [5]. He considers the germs of three holomorphic functions f, g, p at the origin of $\mathbb{C}^n, \mathbb{C}^m$ and $\mathbb{C}^2$ respectively, and then draws conclusions about the topology of the Milnor fiber [4] of $p(f, g)$. Némethi discovered an expression for the Weil zeta function of $p(f, g)$ in terms of the monodromy representations of f and g as well as the several variable Alexander polynomial of p [6, 7].

In this paper we will investigate the case when f and g are homogeneous polynomials and $p(x, y) = xy$. We will construct a fibration different from those of [11, Theorem 2], and [5, 6, 8]. We will then produce a formula for the eigenspace decomposition of the Picard-Lefschetz monodromy of fg in terms of those of f and g.

The following describes the situation we consider,

Hypothesis 1.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $g \in \mathbb{C}[y_1, \ldots, y_m]$ be homogeneous and reduced of positive degrees r and s respectively.

In this case the restriction $f : \mathbb{C}^n \setminus \{x \mid f(x) = 0\} \to \mathbb{C}^*$ defines a fibration. The fiber of this fibration is called the Milnor fiber of f, denoted $F_f = f^{-1}(1)$. Lifting the path $t \mapsto \exp(2\pi i t) : 0 \leq t \leq 1$ in $\mathbb{C}^*$ induces, through a local trivialization of f, a diffeomorphism of the fiber. This map will be called a *geometric* Picard-Lefschetz(PL) monodromy of f. A geometric PL monodromy of f induces a map on the cohomology algebra $H^*(F_f, \mathbb{C})$ which we will call the *algebraic* PL monodromy of f. In a more general setting we may have any smooth fibration $F \to M \to N$. Given a loop in the base space N we may again construct a diffeomorphism of the fiber; in the case when this diffeomorphism is homotopic to the identity for any loop we choose, we say that the fibration has *trivial* geometric monodromy.

For any $M \subseteq \mathbb{C}^t \setminus \{0\}$ we denote by M^* the image of M under the Hopf fibration

$$\rho : \mathbb{C}^t \setminus \{0\} \to \mathbb{P}^{t-1}.$$

Note that as f is homogeneous, F_f^* is the complement of the projective hypersurface defined by $f = 0$. Let $f = f_1 \cdots f_e$ be a factorization of f into irreducible polynomials. We note that $(\mathbb{C}^n \setminus f^{-1}(0))^*$ has first homology generated by the

meridian circles γ_i around $f_i^{-1}(0) \subset \mathbb{P}^{n-1}$ with orientations determined by the complex orientations. For $\eta^r = 1$ we denote by $\mathcal{V}_\eta^f$ the rank one local system on $(\mathbb{C} \setminus f^{-1}(0))^* = F_f^*$ induced by the homomorphism

$$\Phi_\eta^f : H_1(F_f^*) \to GL_1(\mathbb{C}) = \mathbb{C}^* \tag{1.1}$$

that sends $H_1(\rho)(\gamma_i)$ to η. We define $\mathcal{W}_\eta$ to be the local system on $\mathbb{C}^*$ induced by the representation that sends the standard generator of $\pi_1(\mathbb{C}^*)$ to $\eta \in \mathbb{C}^*$. We will also let $H^*(F_f, \mathbb{C})_\eta$ denote the eigenspace of the algebraic PL monodromy of f with eigenvalue η. Lastly, the symbol # will be used as a subscript of a continuous function, and denotes the induced homomorphism defined on the fundamental groups.

We will show the following two results:

Lemma 1.2. *Let f and g be as in Hypothesis 1.1. Then there is a fibration $F_{fg}^* \to F_f^* \times F_g^*$ defined by*

$$[x_1 : \ldots : x_n : y_1 : \ldots : y_m] \mapsto ([x_1 : \ldots : x_n], [y_1 : \ldots : y_m])$$

with fiber $\mathbb{C}^*$ and trivial geometric monodromy.

The Leray spectral sequence associated to this fibration will allow us to prove the following formula.

Theorem 1.3. *Let f and g satisfy Hypothesis 1.1. Then,*

$$H^*(F_{fg}, \mathbb{C})_\eta = H^*(F_f, \mathbb{C})_\eta \otimes H^*(F_g, \mathbb{C})_\eta \otimes H^*(\mathbb{C}^*, \mathbb{C}).$$

In the statement above, the tensor symbol is used to mean the tensor product of vectorspaces graded by cohomological degree. Namely, if M_* and N_* are graded vectorspaces then

$$(M \otimes N)_k = \bigoplus_{i+j=k} M_i \otimes N_j.$$

It clearly follows from Theorem 1.3 that the Weil zeta function of the algebraic PL monodromy of fg is always 1, recovering Némethi's result in our case [5].

2. Proof of Theorem 1.3

We will make heavy use of a Theorem of D.Cohen and A. Suciu [1] that we state here.

Theorem 2.1 (Cohen, Suciu)**.** *Let f be homogeneous and reduced of degree r, and pick $\eta \in \mathbb{C}^*, \eta^r = 1$. Then*

$$H^*(F_f, \mathbb{C})_\eta \cong H^*(F_f^*, \mathcal{V}_\eta^f).$$

□

The equation above simply tells us that an eigenspace of the algebraic PL monodromy of f is isomorphic to the cohomology of a local system defined on the complement of the projective hypersurface $f = 0$. This Theorem leads us to consider the cohomology $H^*(F_{fg}, \mathcal{V}_\eta^{fg})$. We establish Lemma 1.2 to aid in the computation of $H^*(F_{fg}, \mathcal{V}_\eta^{fg})$.

Lemma 2.2. *Let $\mathbb{P}^{n+m-1}$ have coordinates $x_1, \dots, x_n, y_1, \dots, y_m$. Let*

$$M = \mathbb{P}^{n+m-1} \setminus (\mathbb{P}^{n-1} \cup \mathbb{P}^{m-1})$$

be the complement of the projective variety defined by the ideal

$$(x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_n).$$

Then the map

$$\phi : M \to \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$$

sending $[x_1 : \dots : x_n : y_1 : \dots : y_m]$ to $([x_1 : \dots : x_n], [y_1 : \dots : y_m])$ makes M a $\mathbb{C}^*$-bundle over $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ with trivial geometric monodromy.

Proof. This map is clearly well-defined. When we restrict to the chart defined by $x_j \neq 0$ (resp. $y_j \neq 0$) then ϕ can be interpreted as the Hopf fibration applied to the y's (resp. x's). This map is clearly surjective and has trivial geometric monodromy as $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ is simply connected. $\square$

Proof of Lemma 1.2. The restriction of ϕ to F^*_{fg} has image $F^*_f \times F^*_g$ and has trivial geometric monodromy. $\square$

When we look at the Leray spectral sequence induced by this fibration we obtain the following result.

Theorem 2.3. *If Hypothesis 1.1 holds then there is a spectral sequence*

$$E_2^{i,j} \Longrightarrow H^{i+j}(F_{fg}, \mathbb{C})_\eta, \tag{2.1}$$

with $E_2^{i,j} = 0$ for $j \neq 0, 1$, and

$$E_2^{i,0} \cong E_2^{i,1} \cong \bigoplus_{j+k=i} H^j(F_f, \mathbb{C})_\eta \otimes H^k(F_g, \mathbb{C})_\eta.$$

Proof. Let M be the complement of $f = 0$ in $\mathbb{C}^n$ and $f : M \to \mathbb{C}^*$ be the Milnor fibration. On page 107 of [1] we have the following commutative diagram with exact rows.

$$\begin{tikzcd} \pi_1(\mathbb{C}^*) \arrow[r, "\iota"] \arrow[d, "\cong"] & \pi_1(M) \arrow[r] \arrow[d, "f_\#"] & \pi_1(M^*) \arrow[d] \\ \mathbb{Z} \arrow[r, "\times r"] & \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd} \tag{2.2}$$

The top row of this diagram is part of the homotopy sequence associated to the Hopf fibration restricted to M. It also follows from [1] that if $f = f_1 \cdots f_e$ is a factorization of f into distinct irreducible polynomials and if a_i is the homotopy class of a meridian around $f_i = 0$ with orientation determined by the complex orientations, then $f_\#(a_i) = 1$ for all $i = 1, \dots, e$. In this way $H_1(M, \mathbb{Z})$ may be identified with the free $\mathbb{Z}$ module with basis given by the homology classes determined by each of the a_i, and $f_* : H_1(M, \mathbb{Z}) \to \mathbb{Z}$ may be identified with the matrix $[1, 1, \dots, 1]^T$. Also note by commutativity that if σ is an appropriate choice of a generator of $\pi_1(\mathbb{C}^*)$, then we have $f_\# \circ \iota(\sigma) = r$. Thus in particular $\Phi^f_\eta([\iota(\sigma)]) = \eta^r$, where $[*]$ denotes "the homology class determined by".

Now recall the fibration from Lemma 1.2:

$$\mathbb{C}^* \xrightarrow{\kappa} F^*_{fg} \xrightarrow{\phi} F^*_f \times F^*_g\ . \tag{2.3}$$

Recall further that we consider $F_f^* \times F_g^*$ as a subset of $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_m$. The open subset $y_1 \neq 0$ of $F_f^* \times F_g^*$ can be thought of as $M^* \times C$ where C is the complement of the hypersurface $g(1, y_2, \dots, y_m) = 0$ in $\mathbb{A}^{m-1}$. In this way, $\phi^{-1}(M^* \times C)$ is $M \times C$ in $\mathbb{A}^{m+n-1}$. In fact $\phi|_{\phi^{-1}(F_f^* \times C)}$ may be identified with the Hopf fibration applied to the x's,

$$\mathbb{C}^* \longrightarrow M \times C \longrightarrow M^* \times C\ . \tag{2.4}$$

Let $U \subseteq C$ be a contractible subset of C and $\psi = \phi|_{\phi^{-1}(F_f^* \times U)}$, and consider the restriction of (2.4),

$$\mathbb{C}^* \xrightarrow{\lambda} M \times U \xrightarrow{\psi} M^* \times U\ .$$

Then by the discussion in the first paragraph of this proof we know that

$$\Phi_\eta^{fg}([\lambda_\#(\sigma)]) = \eta^r, \tag{2.5}$$

where $[*]$ denotes the image of the homology class of $*$ under the natural map $H^1(F_f^* \times U, \mathbb{Z}) \to H^1(F_f^* \times F_g^*, \mathbb{Z})$.

Let $\eta^{r+s} = 1$. We will see that the spectral sequence of the Theorem is essentially the Leray spectral sequence

$$H^i(F_f^* \times F_g^*, \mathbb{R}^j \phi_*(\mathcal{V}_\eta^{fg})) \Longrightarrow H^{i+j}(F_{fg}^*, \mathcal{V}_\eta^{fg}). \tag{2.6}$$

To compute $\mathbb{R}^\ell \phi_*(\mathcal{V}_\eta^{fg})$, we may apply [2, Proposition 6.4.3] to obtain:

- $\mathbb{R}^\ell \psi_*(\mathcal{V}_\eta^{fg}|_{F_f^* \times U}) = \pi_1^*(\mathcal{V}_\eta^f)|_{F_f^* \times U}$ if $\eta^r = 1$ and $\ell = 0, 1$.
- $\mathbb{R}^\ell \psi_*(\mathcal{V}_\eta^{fg}|_{F_f^* \times U}) = 0$ otherwise.

Here $\pi_1 : F_f^* \times F_g^* \to F_f^*$, and $\pi_2 : F_f^* \times F_g^* \to F_g^*$ are the natural projections. Note that the equation (2.5) calculates what A. Dimca calls the total monodromy operator on [2, p. 210]. A symmetric argument holds with f replaced by g (note that $\eta^r = \eta^{-s}$ as $\eta^{r+s} = 1$) and we may conclude:

- $\mathbb{R}^\ell \phi_*(\mathcal{V}_\eta^{fg}) = \pi_1^*(\mathcal{V}_\eta^f) \otimes \pi_2^*(\mathcal{V}_\eta^g) := \mathcal{V}_\eta^f \boxtimes \mathcal{V}_\eta^g$ if $\eta^r = 1$ and $\ell = 0, 1$.
- $\mathbb{R}^\ell \phi_*(\mathcal{V}_\eta^{fg}) = 0$ otherwise.

Therefore, if $\eta^r \neq 1, \eta^{r+s} = 1$ the spectral sequence (2.6) is zero. Hence $H^*(F_{fg}, \mathbb{C})_\eta = 0$. In this case $H^*(F_f, \mathbb{C})_\eta$ is also zero and the Theorem is proved. When $\eta^r = 1$ we may now apply the Künneth formula [2, Theorem 4.3.14] to obtain that for $j = 0, 1$ one has

$$H^\ell(F_f^* \times F_g^*, \mathbb{R}^j \phi_* \mathcal{V}_\eta^{fg}) = \bigoplus_{i+k=\ell} H^i(F_f^*, \mathcal{V}_\eta^f) \otimes H^k(F_g^*, \mathcal{V}_\eta^f),$$

while the left hand side is zero for other j. These are exactly the $E_2^{\ell,j}$ terms of the spectral sequence of our Theorem, and we know that it converges to $H^*(F_{fg}^*, \mathcal{V}_\eta^{fg}) \cong H^*(F_{fg}^*, \mathbb{C})_\eta$. This establishes the Theorem for $\eta^{r+s} = 1$.

It may be noted that when $\eta^{r+s} \neq 1$ then either η^r or η^s is not equal to one. If $\eta^r \neq 1$ then $H^*(F_f, \mathbb{C})_\eta$ is zero as well as $H^*(F_{fg}, \mathbb{C})_\eta$. If $\eta^s \neq 1$ then $H^*(F_g, \mathbb{C})_\eta$ is zero as well as $H(F_{fg}, \mathbb{C})_\eta$ and the Theorem follows in these cases. ☐

We will now concern ourselves with computing $H^i(F_{fg}, \mathbb{C})$. We first consider a variant of [8, Theorem 4] and [11, Theorem 2] in our homogeneous case. We tacitly use the embedding $F_{fg} \subset \mathbb{C}^n \times \mathbb{C}^m$ in the following statement.

Lemma 2.4. *The map $\widehat{f} : F_{fg} \to \mathbb{C}^*$ defined by $(x, y) \mapsto f(x)$ is a fibration with fiber $F_f \times F_g$. A geometric PL monodromy of this fibration is,*

$$(x, y) \mapsto \left(\exp\left(\frac{2\pi i}{r}\right) x, \exp\left(-\frac{2\pi i}{s}\right) y\right).$$

Proof. Since f and g are homogeneous we have $f(\exp(\frac{t}{r})x) = \exp(t)f(x)$ and $g(\exp(\frac{t}{s})y) = \exp(t)g(y)$. We also note that if $f(x)g(y) = 1$ then $f(x) = g(y)^{-1}$. These two properties prove the Lemma. □

We have a direct consequence, included here for completeness.

Theorem 2.5. *Let f and g satisfy Hypothesis 1.1, then*

$$H^\ell(F_{fg}, \mathbb{C}) \cong \bigoplus_{\lambda=\ell-1}^{\ell} \left(\bigoplus_{\substack{\eta^{r+s}=1 \\ i+j=\lambda}} H^i(F_f, \mathbb{C})_\eta \otimes H^j(F_g, \mathbb{C})_{\eta^{-1}} \right)$$

where the inner sum runs over all possible η.

Proof. The algebraic PL monodromy operator of $\widehat{f}$ is

$$T_f \otimes T_g^{-1} : H^*(F_f, \mathbb{C}) \otimes H^*(F_g, \mathbb{C}) \to H^*(F_f, \mathbb{C}) \otimes H^*(F_g, \mathbb{C})$$

where T_f, T_g are the algebraic PL monodromy operators of the respective fibers. Since T_f, T_g are of finite order, they are diagonalizable. We let $\{a_i\}$ (resp. $\{b_j\}$) be a homogeneous basis of $H^*(F_f, \mathbb{C})$ (resp. $H^*(F_g, \mathbb{C})$) that are eigenvectors of T_f (resp. T_g), with eigenvalue α_i (resp. β_j). In such a case $\{a_i \otimes b_j\}$ are a basis of eigenvectors for $T_f \otimes T_g^{-1}$ with eigenvalue $\alpha_i \beta_j^{-1}$. This shows that

$$\mathbb{R}^\ell \widehat{f}_*(\mathbb{C}_{F_{fg}}) = \bigoplus_{\substack{(\alpha_i, \beta_j) \\ \deg(a_i)+\deg(b_j)=\ell}} \mathcal{W}_{\alpha_i \beta_j^{-1}}.$$

Ergo, since non-constant rank one local systems on $\mathbb{C}^*$ have no cohomology

$$H^p(\mathbb{C}^*, \mathbb{R}^\ell \psi_*(\mathbb{C}_{F_{fg}})) = \bigoplus_{\substack{\alpha_i \beta_j^{-1}=1 \\ \deg(a_i)+\deg(b_j)=\ell}} \mathbb{C}$$

for $p = 0, 1$, and the left hand side is zero for $p \neq 0, 1$.

Now we may consider the Leray spectral sequence associated with the fibration of Lemma 2.4. Since the base of this fibration is $\mathbb{C}^*$, the spectral sequence has only two columns, and thus converges on the second page. This yields the Theorem. □

Proof of Theorem 1.3. Theorem 1.3 now follows from Theorems 2.3 and 2.5. To see this we denote by Γ_η^ℓ the vector space $\bigoplus_{i+j=\ell} H^i(F_f, \mathbb{C})_\eta \otimes H^j(F_g, \mathbb{C})_\eta$. Theorem 2.3 shows the existence of the following exact sequence,

$$\Gamma_\eta^{\ell-2} \xrightarrow{d_2} \Gamma_\eta^\ell \longrightarrow H^\ell(F_{fg}, \mathbb{C})_\eta \longrightarrow \Gamma_\eta^{\ell-1} \xrightarrow{d_2} \Gamma_\eta^{\ell+1}.$$

By [1, Proposition 1.1], $H^\ell(F_f, \mathbb{C})_\eta \cong H^\ell(F_f, \mathbb{C})_{\eta^{-1}}$. Hence

$$H^\ell(F_{fg}, \mathbb{C}) \cong \bigoplus_{\substack{j=\ell-1 \\ \eta^{r+s}=1}}^{\ell} \Gamma_\eta^j,$$

by Theorem 2.5, and so the second differential, d_2 is zero. $\square$

3. A Few Examples

The reader may wish to consult [10] for definitions of terms that involve hyperplane arrangements.

Example 3.1. Let $r = 4, s = 5, f = x_1x_2(x_1 + x_2)(x_1 + 2x_2), g = y_1y_2(y_1 + y_2)(y_1 + 2y_2)(y_1+3y_2)$. Note that f and g define generic central line arrangements. The Weil zeta function of any generic hyperplane arrangement singularity is presented in [9]. To compute the Weil zeta function of any hyperplane arrangement singularity one may use [4, Theorem 9.6], the formula $\chi(F_h) = deg(h)\chi(F_h^*)$, and the algorithm [10, Theorem 5.87(c)]. This method is practical for low dimensions and is simple for line arrangements. Also the Weil zeta function of any generic hyperplane arrangement singularity is presented in [9]. Since the Milnor fiber of f and g is connected we may easily write down tables expressing the eigenspace decomposition of the algebraic (PL) monodromy as follows. Note that we express $\exp(2\pi i/5)$ as ω.

$\dim(H^j(F_f, \mathbb{C})_\eta)$

$\eta\backslash j$	0	1
1	1	3
i	0	2
-1	0	2
$-i$	0	2

$\dim(H^j(F_g, \mathbb{C})_\eta)$

$\eta\backslash j$	0	1
1	1	4
ω	0	3
ω^2	0	3
ω^3	0	3
ω^4	0	3

Now our Theorem 1.3 immediately yields the following table for $\dim(H^j(F_{fg}, \mathbb{C})_\eta)$:

$\eta\backslash j$	0	1	2	3
1	1	8	19	12

where there is a zero for every other η and j. In this example even though the algebraic PL monodromy of f and g have non-unity eigenvalues the algebraic PL monodromy of fg has one as the only eigenvalue.

This behavior is not uncommon. Theorem 1.3 guarantees that $H^*(F_f, \mathbb{C})_\eta$ will not contribute to $H^*(F_{fg}, \mathbb{C})$ if $H^*(F_g, \mathbb{C})_\eta$ is zero. We state this observation as the following Corollary of Theorem 1.3.

Corollary 3.2. *We assume the conditions of Hypothesis 1.1. $H^*(F_{fg}, \mathbb{C})_\eta \neq 0$ if and only if $H^*(F_f, \mathbb{C})_\eta \neq 0$ and $H^*(F_g, \mathbb{C})_\eta \neq 0$. In particular if $H^*(F_{fg}, \mathbb{C})_\eta \neq 0$ then $\eta^{\gcd(r,s)} = 1$.* $\square$

Here we give an if and only if condition for the vanishing of $H^j(F_{fg}, \mathbb{C})_\eta$. The second paragraph of remark 3.2 of [3] only provides the second sentence of this Corollary.

Example 3.3. Let $r = 3, s = 6, f = x_1^3+x_2^3+x_3^3$ and $g = y_1y_2(y_1+y_2)(y_1+2y_2)(y_1+3y_2)(y_1 + 4y_2)$. The eigenspace decomposition of f is discussed in [4, section 9] and g is a line arrangement so we obtain

$\dim(H^j(F_f,\mathbb{C})_\eta)$

$\eta\backslash j$	0	1	2
1	1	0	2
ω^2	0	0	3
ω^4	0	0	3

$\dim(H^j(F_g,\mathbb{C})_\eta)$

$\eta\backslash j$	0	1
1	1	5
ω	0	4
ω^2	0	4
ω^3	0	4
ω^4	0	4
ω^5	0	4

where $\omega = \exp(2\pi i/6)$. Applying our Theorem 1.3 yields the following table for $\dim(H^j(F_{fg},\mathbb{C})_\eta)$.

$\eta\backslash j$	0	1	2	3	4
1	1	6	7	12	10
ω^2	0	0	0	12	12
ω^4	0	0	0	12	12

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